

Fly-automata, model-checking and recognizability

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Abstract

The *Recognizability Theorem* states that if a set of finite graphs is definable by a monadic second-order (MSO) sentence, then it is recognizable with respect to the graph algebra upon which the definition of clique-width is based. Recognizability is an algebraic notion, defined in terms of congruences that can also be formulated by means of finite automata on the terms that describe the considered graphs.

This theorem entails that the verification of MSO graph properties, or equivalently, the model-checking problem for MSO logic over finite binary relational structures, is fixed-parameter tractable (FPT) for the parameter consisting of the formula that expresses the property and the clique-width (or the tree-width) of the input graph or structure. The corresponding algorithms can be implemented by means of *fly-automata* whose transitions are computed on the fly and not tabulated.

We review two versions of recognizability, we present fly-automata by means of examples showing that they can also compute values attached to graphs. We show that *fly-automata with infinite sets* of states yield a simple proof of the strong version of the Recognizability Theorem. This proof has not been published previously.

Keywords: Model-checking; monadic second-order logic; tree-width; clique-width; fixed parameter tractable algorithm; automaton on terms; fly-automaton; recognizability.

Introduction

The *Recognizability Theorem* states that, if a set of finite graphs is definable by a monadic second-order (MSO) sentence, then it is recognizable with respect to the graph algebra upon which the definition of *clique-width* is based. It states a similar result for graphs of bounded tree-width and the corresponding graph algebra [6]. Recognizability is defined algebraically in terms of congruences and can also be formulated by means of finite, or even infinite, automata on the finite terms that describe the considered graphs. Together with other results (see Chapter 6 of [6]), this theorem entails that the verification of MSO graph

properties, or equivalently, the model-checking problem for MSO logic over finite binary relational structures, is fixed-parameter tractable (FPT) for the parameter consisting of the formula that expresses the property and the clique-width of the input graph or structure. Tree-width can also be used instead of clique-width.

Tree-width and *clique-width* are graph complexity measures that serve as parameters in many FPT algorithms [7, 8, 10] and are based on hierarchical decompositions of graphs. These decompositions can be expressed by terms written with the operation symbols of appropriate graph algebras [6]. Model-checking algorithms can be based on automata taking such terms as input. However, the automata associated with MSO formulas, even if they are built for small bounds on tree-width or clique-width, are in practice much too large to be constructed [11, 14]. A typical number of states is $2^{2^{10}}$ and lower-bounds match this number.

We overcome this difficulty by using *fly-automata* (FA) [3]. They are automata whose states are *described* and not *listed*, and whose transitions are *computed on the fly* and not *tabulated*. When running on a term of size 1000, a deterministic FA with $2^{2^{10}}$ states computes only 1000 transitions.

Fly-automata can have infinitely many states. For example, a state can record, among other things, the (unbounded) number of occurrences of a particular symbol in the input term. FA can thus check some graph properties that are *not monadic second-order expressible*. An example is *regularity*, the fact that all vertices have the same degree. Furthermore, an FA equipped with an *output function* that maps the set of accepting states to an effectively given domain \mathcal{D} can compute a value, for example the number of k -colorings of the given graph G , or the minimum cardinality of one of the k color classes if G is k -colorable (this number measures how close is this graph to be $(k - 1)$ -colorable). We have computed with FA the numbers of k -colorings for $k = 3, 4, 5$ of some graphs (cycles, trees, Petersen graph) for which the chromatic polynomial is known, so that we could test the correctness of the automata (their correctness can anyway be proved formally).

In this article, we review recognizability, fly-automata and their applications to the verification of properties or the computation of values associated with graphs. We present results concerning graphs of bounded clique-width. Similar results for graphs of bounded tree-width reduce to them as we will explain. In an appendix that can be read as an addendum to [3], we explain how the Recognizability Theorem can be proved by means of fly-automata, in an easier way than in Chapter 5 of [6].

1 Graph algebras, recognizability and automata

Graphs are finite, undirected, without loops and multiple edges. The extension to directed graphs, possibly with loops and/or labels is straightforward. A graph G is identified with the relational structure $\langle V_G, \text{edg}_G \rangle$ where edg_G is a binary symmetric relation representing adjacency.

Rather than giving a formal definition of *monadic second-order* (MSO) logic, we present the sentence (i.e., the formula without free variables) expressing 3-colorability (an NP-complete property). It is $\exists X, Y. \text{Col}(X, Y)$ where $\text{Col}(X, Y)$ is the formula

$$\begin{aligned} X \cap Y = \emptyset \wedge \forall u, v. \{ \text{edg}(u, v) \implies \\ [\neg(u \in X \wedge v \in X) \wedge \neg(u \in Y \wedge v \in Y) \wedge \\ \neg(u \notin X \cup Y \wedge v \notin X \cup Y)] \}. \end{aligned}$$

This formula expresses that X, Y and $V_G - (X \cup Y)$ are the three color classes of a 3-coloring.

Definition 1 : *The graph algebra \mathcal{G}*

(a) We will use \mathbb{N}_+ as a set of labels called *port labels*. A *p-graph* is a triple $G = \langle V_G, \text{edg}_G, \pi_G \rangle$ where π_G is a mapping : $V_G \rightarrow \mathbb{N}_+$. If $\pi_G(x) = a$, we say that x is an *a-port*. The set $\pi(G)$ of port labels of G is its *type*. By using a default label, say 1, we make every nonempty graph into a p-graph of type $\{1\}$.

(b) For each $k \in \mathbb{N}_+$, we define a finite set F_k of *operations* on p-graphs of type included in $C = \{1, \dots, k\}$ that consists of :

- the binary symbol \oplus denotes the union of two *disjoint* p-graphs,
- the unary symbol $\text{relab}_{a \rightarrow b}$ denotes the *relabelling* that changes every port label a into b (where $a, b \in C$),
- the unary symbol $\text{add}_{a,b}$, for $a < b$, $a, b \in C$, denotes the *edge-addition* that adds an edge between every a -port and every b -port (unless there is already an edge between them; our graphs have no multiple edges),
- for each $a \in C$, the nullary symbol \mathbf{a} denotes an isolated a -port.

(c) Every term t in $T(F_k)$ (the set of finite terms written with F_k) is called a *k-expression*. Its *value* is a p-graph, $\text{val}(t)$, that we now define. We denote by $\text{Pos}(t)$ the set of *positions* of t : they are the nodes of the syntactic tree of t and the occurrences of symbols. For each $u \in \text{Pos}(t)$, we define a p-graph $\text{val}(t)/u$, whose vertex set is the set of leaves of t below u . The definition of $\text{val}(t)/u$ is, for a fixed t , by bottom-up induction on u :

- if u is an occurrence of \mathbf{a} , then $\text{val}(t)/u$ has vertex u as an a -port and no edge,
- if u is an occurrence of \oplus with sons u_1 and u_2 , then

$val(t)/u := val(t)/u_1 \oplus val(t)/u_2$, (note that $val(t)/u_1$ and $val(t)/u_2$ are disjoint),

- if u is an occurrence of $relab_{a \rightarrow b}$ with son u_1 , then

$val(t)/u := relab_{a \rightarrow b}(val(t)/u_1)$,

- if u is an occurrence of $add_{a,b}$ with son u_1 , then

$val(t)/u := add_{a,b}(val(t)/u_1)$.

Finally, $val(t) := val(t)/root_t$. Note that its vertex set is the set of all leaves (occurrences of nullary symbols). For an example, let

$$t = add_{b,c}^1(add_{a,b}^2(\mathbf{a}^3 \oplus^4 \mathbf{b}^5) \oplus^6 relab_{b \rightarrow c}^7(add_{a,b}^8(\mathbf{a}^9 \oplus^{10} \mathbf{b}^{11})))$$

where the superscripts 1 to 11 number the positions of t . The p-graph $val(t)$ is $3_a - 5_b - 11_c - 9_a$ where the subscripts a, b, c indicate the port labels. (For clarity, port labels are letters in examples). If $u = 2$ and $w = 8$, then $t/u = t/w = add_{a,b}(\mathbf{a} \oplus \mathbf{b})$, however, $val(t)/u$ is the p-graph $3_a - 5_b$ and $val(t)/w$ is $9_a - 11_b$, isomorphic to $val(t)/u$.

(d) The *clique-width* of a graph G , denoted by $cwd(G)$, is the least integer k such that G is isomorphic to $val(t)$ for some t in $T(F_k)$. We denote by \mathcal{G}_k the set $val(T(F_k))$ of p-graphs that are the value of a term over F_k . We let F be the union of the sets F_k , and \mathcal{G} be the union of the sets \mathcal{G}_k . Every p-graph is isomorphic to a graph in \mathcal{G} , hence, is defined by some term hence, has a well-defined clique-width.

(e) An *F-congruence* is an equivalence relation \approx on p-graphs such that :

- two isomorphic p-graphs are equivalent, and
- if $G \approx G'$ and $H \approx H'$, then $\pi(G) = \pi(G')$, $add_{a,b}(G) \approx add_{a,b}(G')$, $relab_{a \rightarrow b}(G) \approx relab_{a \rightarrow b}(G')$ and $G \oplus H \approx G' \oplus H'$.

(f) A set of graphs L is *recognizable* if it is a (possibly infinite) union of classes of an *F-congruence* that has finitely many classes of each finite type $C \subseteq \mathbb{N}_+$.

Definition 2: *Fly-automata.*

(a) Let H be a finite or countable, effectively given, signature with arity mapping denoted by ρ . A *fly-automaton over H* (in short, an FA *over H*)¹ is a 4-tuple $\mathcal{A} = \langle H, Q_{\mathcal{A}}, \delta_{\mathcal{A}}, Acc_{\mathcal{A}} \rangle$ such that $Q_{\mathcal{A}}$ is the finite or countable, effectively given set of *states*, $Acc_{\mathcal{A}}$ is the set of *accepting states*, a decidable subset of $Q_{\mathcal{A}}$, and $\delta_{\mathcal{A}}$ is a computable function that defines the *transition rules*: for each tuple (f, q_1, \dots, q_m) such that $q_1, \dots, q_m \in Q_{\mathcal{A}}$, $f \in H$, $\rho(f) = m \geq 0$, $\delta_{\mathcal{A}}(f, q_1, \dots, q_m)$ is a finite set of states. We write $f[q_1, \dots, q_m] \rightarrow q$ (and $f \rightarrow q$ if f is nullary) to mean that $q \in \delta_{\mathcal{A}}(f, q_1, \dots, q_m)$. We say that \mathcal{A} is *finite* if F and $Q_{\mathcal{A}}$ are finite. Even in this case, it is interesting to have these sets *specified*

¹A fly-automaton is an automaton on finite terms whose components are finite or countably infinite and effectively given, and that has finitely many runs on each term.

rather than listed because this allows to implement finite automata with huge sets of states [3–5].

(b) A *run* of \mathcal{A} on a term $t \in T(H)$ is a mapping $r : Pos(t) \rightarrow Q_{\mathcal{A}}$ such that:

if $u \in Pos(t)$ is an occurrence of f with sequence of sons u_1, \dots, u_m ,
then $r(u) \in \delta_{\mathcal{A}}(f, r(u_1), \dots, r(u_m))$.

A run r is *accepting* if $r(\text{root}_t) \in Acc_{\mathcal{A}}$. A term t is *accepted* (or *recognized*) by \mathcal{A} if it has an accepting run. We denote by $L(\mathcal{A})$ the set of terms accepted by \mathcal{A} . A *deterministic* FA \mathcal{A} (by "deterministic" we mean "deterministic and complete") has a unique run on each term t and $q_{\mathcal{A}}(t)$ is the state reached at the root of t . The mapping $q_{\mathcal{A}}$ is computable and the membership in $L(\mathcal{A})$ of a term $t \in T(H)$ is decidable.

(c) Every FA \mathcal{A} that is not deterministic can be *determinized* by an easy extension of the usual construction, see [3]; it is important that the sets $\delta_{\mathcal{A}}(f, q_1, \dots, q_m)$ be finite.

(d) A deterministic FA over H with *output function* is a 4-tuple $\mathcal{A} = \langle H, Q_{\mathcal{A}}, \delta_{\mathcal{A}}, Out_{\mathcal{A}} \rangle$ that is a deterministic FA where $Acc_{\mathcal{A}}$ is replaced by a total and computable *output function* $Out_{\mathcal{A}} : Q_{\mathcal{A}} \rightarrow \mathcal{D}$ such that \mathcal{D} is an effectively given domain. The *function computed by* \mathcal{A} is $Comp(\mathcal{A}) : T(H) \rightarrow \mathcal{D}$ such that $Comp(\mathcal{A})(t) := Out_{\mathcal{A}}(q_{\mathcal{A}}(t))$.

Example 1: The number of accepting runs of an automaton.

Let $\mathcal{A} = \langle H, Q_{\mathcal{A}}, \delta_{\mathcal{A}}, Acc_{\mathcal{A}} \rangle$ be a nondeterministic FA. We construct a deterministic FA \mathcal{B} that computes the number of accepting runs of \mathcal{A} on any term in $T(H)$. The set of states is the set of finite subsets of $Q_{\mathcal{A}} \times \mathbb{N}_+$. The transitions are defined so that \mathcal{B} reaches state α at the root of $t \in T(H)$ if and only if α is the finite set of pairs $(q, n) \in Q_{\mathcal{A}} \times \mathbb{N}_+$ such that n is the number of runs of \mathcal{A} that reach state q at the root. This number is finite and α can be seen as a partial function $Q_{\mathcal{A}} \rightarrow \mathbb{N}_+$ having a finite domain. For a symbol f of arity 2, \mathcal{B} has the transition $f[\alpha, \beta] \rightarrow \gamma$ where γ is the set of pairs (q, n) such that $q \in Q_{\mathcal{A}}$ and n is the sum of the integers $n_p \times n_r$ such that $(p, n_p) \in \alpha$ and $(r, n_r) \in \beta$. The transitions for other symbols are defined similarly. The function $Out_{\mathcal{A}}$ maps a state α to the sum of the integers n such that $(q, n) \in \alpha \cap (Acc_{\mathcal{A}} \times \mathbb{N}_+)$. \square

Example 2: A fly-automaton that checks 3-colorability.

In order to construct an FA that accepts the terms $t \in T(F)$ such that the graph $val(t)$ is 3-colorable, we first construct an FA \mathcal{A} for the property $Col(X, Y)$, taking two sets of vertices X and Y as arguments. For this purpose, we transform the signature F into $F^{(2)}$ by replacing each nullary symbol \mathbf{a} by the four nullary symbols (\mathbf{a}, ij) , $i, j \in \{0, 1\}$. A term $t \in T(F^{(2)})$ defines two things: first, the graph $val(t')$ where t' is obtained from t by removing the Booleans i, j from the nullary symbols and, second, the pair (X, Y) such that X is the set of vertices u (leaves of t) that are occurrences of $(\mathbf{a}, 1j)$ for some \mathbf{a}

and j , and Y is the set of those that are occurrences of $(\mathbf{a}, i1)$ for some \mathbf{a} and i . The set of terms $t \in T(F^{(2)})$ such that $Col(X, Y)$ holds in $val(t')$ is defined by a deterministic FA \mathcal{A} than we now specify. The coloring defined by X, Y assigns colors 1,2,3 to the vertices respectively in X , Y and $V_G - (X \cup Y)$. Its *type* is the set of pairs (a, i) such that $val(t')$ has an a -port of color i .

We now describe the meaning of the states of \mathcal{A} . If $u \in Pos(t)$ then V_u is the set of vertices of $val(t')/u$, i.e., of leaves below u . At position u of t , the automaton \mathcal{A} reaches the state *Error* if and only if $X \cap Y \cap V_u \neq \emptyset$ or $val(t')/u$ has an edge between two vertices, either both in $X \cap V_u$, or both in $Y \cap V_u$, or both in $V_u - (X \cup Y)$, hence of same color, respectively 1,2 or 3; otherwise, $X \cap V_u$ and $Y \cap V_u$ define a 3-coloring of $val(t')/u$ and \mathcal{A} reaches state $\alpha \subseteq C \times \{1, 2, 3\}$ where α is the type of this coloring. All states except *Error* are accepting. Here are the transitions of \mathcal{A} :

$$\begin{aligned} (\mathbf{a}, 00) &\rightarrow \{(a, 3)\}, (\mathbf{a}, 10) \rightarrow \{(a, 1)\}, (\mathbf{a}, 01) \rightarrow \{(a, 2)\}, \\ (\mathbf{a}, 11) &\rightarrow Error. \end{aligned}$$

For $\alpha, \beta \subseteq C \times \{1, 2, 3\}$, \mathcal{A} has transitions :

$$\begin{aligned} \oplus[\alpha, \beta] &\rightarrow \alpha \cup \beta, \\ add_{a,b}[\alpha] &\rightarrow Error, \text{ if } (a, i) \text{ and } (b, i) \text{ belong to } \alpha \text{ for some } i = 1, 2, 3, \\ add_{a,b}[\alpha] &\rightarrow \alpha, \text{ otherwise,} \\ relab_{a \rightarrow b}[\alpha] &\rightarrow \beta, \text{ obtained by replacing } a \text{ by } b \text{ in each pair of } \alpha. \end{aligned}$$

Its other transitions are $\oplus[\alpha, \beta] \rightarrow Error$ if α or β is *Error*, $add_{a,b}[Error] \rightarrow Error$ and $relab_{a \rightarrow b}[Error] \rightarrow Error$.

This FA checks $Col(X, Y)$. To check, $\exists X, Y. Col(X, Y)$, we build a *nondeterministic* FA \mathcal{B} by deleting the state *Error* and the rules containing *Error*, and by replacing the first three rules of \mathcal{A} by $\mathbf{a} \rightarrow \{(a, 3)\}$, $\mathbf{a} \rightarrow \{(a, 1)\}$, $\mathbf{a} \rightarrow \{(a, 2)\}$. All states are accepting but on some terms, no run of \mathcal{B} can reach the root, and these terms are rejected. Furthermore, the construction of Example 1 shows how to make \mathcal{B} into a deterministic FA that computes the number of accepting runs of \mathcal{B} on a term t , hence of 3-colorings of the graph $val(t)$ because its colorings are in bijection with the accepting runs of \mathcal{B} on t . \square

Example 3: Minimal use of one color.

Continuing Example 2, we want to compute the minimal cardinality of a set X such that $Col(X, Y)$ holds for some set Y . This cardinality is ∞ if the considered graph is not 3-colorable. It is 0 if it is 2-colorable. We build from \mathcal{A} a deterministic FA \mathcal{A}' over $F^{(2)}$ whose states are *Error* and the pairs $(\alpha, m) \in \mathcal{P}(C \times \{1, 2, 3\}) \times \mathbb{N}$. ($\mathcal{P}(X)$ denotes the powerset of a set X .) The meanings of these states are as for \mathcal{A} except that m in (α, m) is the cardinality of $X \cap V_u$. Some rules of \mathcal{A}' are:

$$(\mathbf{a}, 00) \rightarrow (\{(a, 3)\}, 0),$$

$$\begin{aligned}
(\mathbf{a}, 10) &\rightarrow (\{(a, 1)\}, 1), \\
(\mathbf{a}, 01) &\rightarrow (\{(a, 2)\}, 0) \\
\text{and } \oplus[(\alpha, m), (\beta, p)] &\rightarrow (\alpha \cup \beta, m + p).
\end{aligned}$$

We make \mathcal{A}' nondeterministic as in Example 2 and we now detail the deterministic FA \mathcal{C} with output function intended to compute the minimal cardinality of X such that $Col(X, Y)$ holds for some set Y .

Its states are finite sets of pairs $(\alpha, m) \in \mathcal{P}(C \times \{1, 2, 3\}) \times \mathbb{N}$. At the root of a term $t \in T(F)$, the FA \mathcal{C} reaches a set $\sigma \subseteq \mathcal{P}(C \times \{1, 2, 3\}) \times \mathbb{N}$ such that :

- for each $\alpha \in \mathcal{P}(C \times \{1, 2, 3\})$ and $m \in \mathbb{N}$, the pair (α, m) is in σ if and only if:
- α is the type of a 3-coloring defined by a pair (X, Y) ,
- and m is the minimal cardinality of a set X in such a pair.

Note that m is uniquely defined from α . A state can be defined as a partial function : $\mathcal{P}(C \times \{1, 2, 3\}) \rightarrow \mathbb{N}$.

The case $\sigma = \emptyset$ corresponds to a graph that is not 3-colorable, hence, \emptyset plays the role of an *Error* state.

The transitions of \mathcal{C} are as follows:

$$\begin{aligned}
\mathbf{a} &\rightarrow \{(\{(a, 3)\}, 0), (\{(a, 1)\}, 1), (\{(a, 2)\}, 0)\}, \\
\oplus[\sigma, \sigma'] &\rightarrow \sigma'' \text{ where } (\gamma, m) \in \sigma'' \text{ if and only if } m \text{ is the minimum} \\
&\text{number } n + n' \text{ such that } (\alpha, n) \in \sigma, (\beta, n') \in \sigma' \text{ and } \alpha \cup \beta = \gamma, \\
add_{a,b}[\sigma] &= \sigma' \text{ where } \sigma' \text{ is obtained from } \sigma \text{ by removing the pairs} \\
&(\alpha, m) \text{ such that } \alpha \text{ contains } (a, i) \text{ and } (b, i) \text{ for some } i = 1, 2, 3, \\
relab_{a \rightarrow b}[\sigma] &= \sigma' \text{ where } \sigma' \text{ is obtained by replacing every pair } (a, i) \\
&\text{occurring in the first component of any } (\alpha, m) \in \sigma \text{ by } (b, i).
\end{aligned}$$

The output function associates with σ the minimal m such that $(\alpha, m) \in \sigma$ for some α . If $\sigma = \emptyset$ the output value is ∞ because the graph $val(t)$ is not 3-colorable.

Remark: To compute the desired value, we could also use the determinized automaton of \mathcal{A}' with an appropriate output function. Its states encode, for each α , the set of cardinalities $|X|$ such that α is the type of a 3-coloring defined by a pair (X, Y) , instead of just the minimal cardinality of such a set. This way, the computation would take more space and more time. \square

The constructions of these three examples are particular cases of systematic and more complex constructions presented in [3–5].

2 Two recognizability theorems

Two theorems relate MSO logic and recognizability.

Recognizability Theorem : The set of graphs that satisfy an MSO sentence φ is F -recognizable.

Weak Recognizability Theorem : For every MSO sentence φ , for every k , the set of graphs in \mathcal{G}_k that satisfy φ is F_k -recognizable.

About proofs: The Recognizability Theorem is Theorem 5.68 of [6]. Its proof shows that the equivalence relation defined by the fact that two p-graphs have the same type and satisfy the same MSO sentences of quantifier-height at most that of φ satisfies the conditions of Definition 1(f). The Weak Recognizability Theorem follows from the former one. It can also be proved directly by constructing, for each φ and k , a finite automaton $\mathcal{A}(\varphi, k)$ (Theorem 6.35 of [6]). One can also construct for each φ a single FA $\mathcal{A}(\varphi)$ over F that can be seen as the union of the automata $\mathcal{A}(\varphi, k)$ ([3]). This construction has been implemented (see below). The proof of the strong theorem in Chapter 5 of [6] does not provide any usable automaton. As explained in Section 6.4.6 of [6], the Recognizability Theorem is not a corollary of its weak form. However, a careful analysis of $\mathcal{A}(\varphi)$ yields a simple proof of the Recognizability Theorem as we show in the Appendix.

3 Other uses of fly-automata

Counting and optimizing automata

Let $P(X_1, \dots, X_s)$ be an MSO property of vertex sets X_1, \dots, X_s . We denote (X_1, \dots, X_s) by \overline{X} and $t \models P(\overline{X})$ means that \overline{X} satisfies P in the graph $val(t)$ defined by a term t . We are interested, not only to check the validity of $\exists \overline{X}.P(\overline{X})$, but also to compute from a term t the following values:

$\#\overline{X}.P(\overline{X})$, defined as the number of assignments \overline{X} such that

$t \models P(\overline{X})$,

$\text{Sp}\overline{X}.P(\overline{X})$, the *spectrum* of $P(\overline{X})$, defined as the set of tuples of the form $(|X_1|, \dots, |X_s|)$ such that $t \models P(\overline{X})$,

$\text{MSp}\overline{X}.P(\overline{X})$, the *multispectrum* of $P(\overline{X})$, defined as the multiset of tuples $(|X_1|, \dots, |X_s|)$ such that $t \models P(\overline{X})$,

the number $\min\{|Y| \mid \exists \overline{X}.P(Y, \overline{X})\}$.

These computations can be done by FA [4, 5]. We obtain in this way fixed-parameter tractable (FPT) or XP algorithms (see [8, 10] for the theory of fixed-parameter tractability). A particular case of the construction for $\#\overline{X}.P(\overline{X})$ is based on Example 1. (In general, the number $\#\overline{X}.P(\overline{X})$ may be larger than

the number of accepting runs of the nondeterministic automaton that checks $\exists \overline{X}.P(\overline{X})$.

Beyond MS logic

The property that the considered graph is the union of two disjoint regular graphs with possibly some edges between these two subgraphs is not MSO expressible but can be checked by an FA. An FA can also compute the minimal number of edges between X and $V_G - X$ such that $G[X]$ and $G[V_G - X]$ are connected, when such a set X exists.

Edge set quantifications and tree-width.

The incidence graph $Inc(G)$ of a graph G (that can have multiple edges) is a bipartite graph whose vertex set is $V_G \cup E_G$ where E_G is the set of edges of G , and $Inc(G)$ has an edge between $x \in V_G$ and $e \in E_G$ if and only if x is an end of e . An MSO sentence evaluated in $Inc(G)$ is thus able to use quantifications on edges and sets of edges. The graph properties expressed by such sentences are said to be MSO_2 expressible. That G is Hamiltonian can be expressed by "there exists a set of edges that forms a Hamiltonian cycle", hence is MSO_2 expressible, whereas this property is not MSO expressible. If G has tree-width k , then $Inc(G)$ has clique-width at most 3.2^k (see [6], Proposition 2.114) and even at most $k + 3$ by a recent unpublished result due to T. Bouvier (LaBRI). Furthermore, a term defining $Inc(G)$ that witnesses the latter upperbound can be obtained in linear time from a tree-decomposition of G of width k . It follows that FA can be used to verify MSO_2 expressible properties of graphs of bounded tree-width. Counting and optimizing functions based on such properties can also be computed by FA. Another approach is in [2].

The two recognizability theorems have versions for MSO_2 expressible properties of graphs of bounded tree-width (see [6], Theorems 5.68 and 5.69).

4 Experimental results and open problems

These constructions have been implemented and tested² [3–5]. We have computed the number of optimal colorings of some graphs of clique-width at most 8 for which the chromatic polynomial is known, which allowed us to verify the correctness of the automaton. We could verify in, respectively, 35 and 105 minutes that the 20×20 and the 6×60 grids are 3-colorable. In 29 minutes, we could verify that the McGee graph (24 vertices) given by a term over F_8 is acyclically 3-colorable.

A different model-checking method based on games is presented in [13]. It gives a proof of the Weak Recognizability Theorem for graphs of bounded tree-width and has also been implemented and tested.

²AUTOGRAPH is written in Steele Bank Common Lisp and computations have been done on a Mac Book Pro (Mac OS X 10.9.4 with processor Intel Core 2 Duo, 2.53 GHz and memory of 4 GB, 1067 MHz DDR3).

The *parsing problem* for graphs of clique-width at most k is NP-complete (with k in the input) [9]. Good heuristics remain to be developed.

References

- [1] H. Comon et al., Tree automata techniques and applications, 2007, <http://tata.gforge.inria.fr/>
- [2] B. Courcelle, On the model-checking of monadic second-order formulas with edge set quantifications, *Discrete Applied Mathematics* **160** (2012) 866-887.
- [3] B. Courcelle and I. Durand, Automata for the verification of monadic second-order graph properties, *J. Applied Logic* **10** (2012) 368-409 (also <http://hal.archives-ouvertes.fr/hal-00611853/fr/>).
- [4] B. Courcelle and I. Durand, Computations by fly-automata beyond monadic second-order logic, submitted for publication, 2013, <http://arxiv.org/abs/1305.7120>.
- [5] B. Courcelle and I. Durand: Model-checking by infinite fly-automata. *Proceedings of the 5th Conference on Algebraic Informatics, Lecture Notes in Computer Science*, vol. **8080** (2013) 211-222.
- [6] B. Courcelle and J. Engelfriet, *Graph structure and monadic second-order logic, a language theoretic approach*, vol. **138** of *Encyclopedia of Mathematics and its Application*, Cambridge University Press, June 2012.
- [7] B. Courcelle, J. Makowsky and U. Rotics, Linear-time solvable optimization problems on graphs of bounded clique-width, *Theory Comput. Syst.* **33** (2000) 125-150.
- [8] R. Downey and M. Fellows, *Parameterized complexity*, Springer-Verlag, 1999.
- [9] M. Fellows, F. Rosamond, U. Rotics and S. Szeider, Clique-Width is NP-Complete, *SIAM Journal on Discrete Mathematics*, **23** (2009) 909-939.
- [10] J. Flum and M. Grohe, *Parametrized complexity theory*, Springer, 2006.
- [11] M. Frick and M. Grohe, The complexity of first-order and monadic second-order logic revisited, *Ann. Pure Appl. Logic* **130** (2004) 3-31.
- [12] M. Grohe and S. Kreutzer, Model theoretic methods in finite combinatorics, in Grohe and Makowsky (eds), *Contemporary Mathematics* **558**, *American Mathematical Society*, 2011, pp. 181-206.
- [13] J. Kneis, A. Langer and P. Rossmanith: Courcelle's theorem - A game-theoretic approach. *Discrete Optimization* **8** (2011) 568-594.

- [14] K. Reinhardt, The complexity of translating logic to finite automata, in *Automata, logics, and infinite games: a guide to current research*, E. Graedel et al. eds., *Lecture Notes in Computer Scienc*, vol. **2500** (2002) 231-238.

Appendix

We explain to the reader familiar with [3] how the Recognizability Theorem can be proved from the construction, for every MSO sentence φ , of an FA $\mathcal{A}(\varphi)$ over F that recognizes the terms whose value is a finite model of φ . This is a new proof of this theorem. We first review definitions and notation.

MSO formulas are written with set variables X_1, \dots, X_n, \dots (without first-order variables), the atomic formulas $X_i \subseteq X_j$, $Sgl(X_i)$ (meaning that X_i is singleton), $edg(X_i, X_j)$ (meaning that X_i and X_j are singletons consisting of adjacent vertices), negation, conjunction and existential quantifications of the form $\exists X_n. \psi$ where ψ has its free variables among X_1, \dots, X_n .

Generalizing the definition of Example 2, we transform F into $F^{(m)}$ (for $m > 0$) by replacing each nullary symbol \mathbf{a} by the nullary symbols (\mathbf{a}, w) for all $w \in \{0, 1\}^m$. Hence, a term $t \in T(F^{(m)})$ defines the p-graph $val(t')$ where t' is obtained from t by removing the sequences w from the nullary symbols and the m -tuple (V_1, \dots, V_m) such that V_i is the set of vertices u (leaves of t) that are occurrences of (\mathbf{a}, w) for some \mathbf{a} and w with 1 at its i -th position. We denote $val(t')$ by $val(t)$ and (V_1, \dots, V_m) by $\nu(t)$.

If φ is an MSO formula with free variables among X_1, \dots, X_m , we let $L(\varphi, X_1, \dots, X_m)$ be the set of terms $t \in T(F^{(m)})$ such that $(val(t), \nu(t)) \models \varphi$.

Theorem [3] : Let φ be an MSO formula with free variables among X_1, \dots, X_m . One can construct a fly-automaton $\mathcal{A}(\varphi, X_1, \dots, X_m)$ over $F^{(m)}$ that recognizes the set $L(\varphi, X_1, \dots, X_m)$.

We revisit this construction to prove the Recognizability Theorem. For a finite set B and an integer $i \geq 0$, we define the finite set $\mathcal{L}_i(B)$ as follows:

$$\begin{aligned}\mathcal{L}_0(B) &:= B, \\ \mathcal{L}_{i+1}(B) &:= \mathcal{L}_i(B) \cup \mathcal{P}(\mathcal{L}_i(B)) \cup \mathcal{L}_i(B) \times \mathcal{L}_i(B).\end{aligned}$$

In order to have a unique notation for the elements of these sets, we write an element of $\mathcal{P}(\mathcal{L}_i(B))$ as $\{w_1, \dots, w_p\}$ with the condition that $w_1 < w_2 < \dots < w_p$ for some lexicographic order $<$ on the words denoting the elements of the sets $\mathcal{L}_n(B)$.

The proof of the previous theorem yields the following more precise statement.

Proposition : Let φ be an MSO formula with free variables among X_1, \dots, X_m . One can construct a finite set B disjoint from \mathbb{N}_+ , an integer i and a

deterministic fly-automaton $\mathcal{A}(\varphi, X_1, \dots, X_m)$ over $F^{(m)}$ that recognizes the set $L(\varphi, X_1, \dots, X_m)$ and satisfies the following two properties, for all $t, t' \in T(F^{(m)})$:

- (i) $q_{\mathcal{A}(\varphi, X_1, \dots, X_m)}(t) \in \mathcal{L}_i(B \cup \pi(\text{val}(t)))$,
- (ii) if $(\text{val}(t), \nu(t))$ is isomorphic to $(\text{val}(t'), \nu(t'))$, then $q_{\mathcal{A}(\varphi, X_1, \dots, X_m)}(t) = q_{\mathcal{A}(\varphi, X_1, \dots, X_m)}(t')$.

Proof : The proof is by induction on the structure of φ . We assume that the reader has access to [3], so we will not detail the automata.

If φ is $X_i \subseteq X_j$ or $\text{Sgl}(X_i)$, then the states of $\mathcal{A}(\varphi, X_1, \dots, X_m)$ do not use port labels, hence $q_{\mathcal{A}(\varphi, X_1, \dots, X_m)}(t) \in \mathcal{L}_0(B)$ for some finite set B . Properties (i) and (ii) hold.

If φ is $\text{edg}(X_i, X_j)$, then the states of $\mathcal{A}(\varphi, X_1, \dots, X_m)$ are *Error*, *Ok*, $(a, \underline{1})$, $(a, \underline{2})$, (a, b) for $a, b \in \mathbb{N}_+$, hence belong to $B \cup \mathbb{N}_+ \times (\mathbb{N}_+ \cup B)$ where $B = \{\text{Error}, \text{Ok}, \underline{1}, \underline{2}\}$. (To be precise the states $(a, \underline{1})$, $(a, \underline{2})$ and (a, b) are written respectively $a(1)$, $a(2)$ and ab in [3].) The port labels occurring in the state $q_{\mathcal{A}(\text{edg}(X_i, X_j), X_1, \dots, X_m)}(t)$ are in $\pi(\text{val}(t))$: this is clear from the meanings of the states described in Table 3 of [3]. So we have $q_{\mathcal{A}(\varphi, X_1, \dots, X_m)}(t) \in \mathcal{L}_1(B \cup \pi(\text{val}(t)))$. The validity of (ii) is also clear from the same table.

If φ is $\neg\psi$ and since we construct deterministic (and complete) automata, $\mathcal{A}(\varphi, X_1, \dots, X_m)$ and $\mathcal{A}(\psi, X_1, \dots, X_m)$ differ only in their accepting states. Hence $\mathcal{A}(\varphi, X_1, \dots, X_m)$ satisfies Properties (i) and (ii) since $\mathcal{A}(\psi, X_1, \dots, X_m)$ does.

If φ is $\theta \wedge \psi$, then $\mathcal{A}(\varphi, X_1, \dots, X_m)$ is the product automaton of $\mathcal{A}(\theta, X_1, \dots, X_m)$ and $\mathcal{A}(\psi, X_1, \dots, X_m)$ (in particular $Q_{\mathcal{A}(\varphi, X_1, \dots, X_m)} = Q_{\mathcal{A}(\theta, X_1, \dots, X_m)} \times Q_{\mathcal{A}(\psi, X_1, \dots, X_m)}$). If (B, i) and (B', j) are associated by induction with θ and ψ , then we can take the pair $(B \cup B', 1 + \max(i, j))$ for φ , which gives Property (i). Property (ii) is easy to check.

If φ is $\exists X_m. \psi$ where ψ has its free variables among X_1, \dots, X_m , then $\mathcal{A}(\varphi, X_1, \dots, X_{m-1})$ is obtained from $\mathcal{A}(\psi, X_1, \dots, X_m)$ as follows:

- (1) one builds an FA \mathcal{B} by replacing in $\mathcal{A}(\psi, X_1, \dots, X_m)$ all transitions $(\mathbf{a}, w0) \rightarrow p$ and $(\mathbf{a}, w1) \rightarrow q$ by $(\mathbf{a}, w) \rightarrow p$ and $(\mathbf{a}, w) \rightarrow q$ so that \mathcal{B} is not deterministic;
- (2) $\mathcal{A}(\varphi, X_1, \dots, X_{m-1})$ is defined as the determinized automaton of \mathcal{B} .

If $q_{\mathcal{A}(\psi, X_1, \dots, X_m)}(t) \in \mathcal{L}_i(B \cup \pi(\text{val}(t)))$, then $q_{\mathcal{A}(\varphi, X_1, \dots, X_{m-1})}(t) \in \mathcal{P}(\mathcal{L}_i(B \cup \pi(\text{val}(t)))) \subseteq \mathcal{L}_{i+1}(B \cup \pi(\text{val}(t)))$, which proves (i). Property (ii) is easy to check.

It may be necessary to construct $\mathcal{A}(\varphi, X_1, \dots, X_m)$ from $\mathcal{A}(\varphi, X_1, \dots, X_n)$ where $m > n$. A typical example is for $\theta = \varphi \wedge \psi$ in a case where we have already constructed $\mathcal{A}(\varphi, X_1, X_2)$ and $\mathcal{A}(\psi, X_1, X_2, X_3)$; we must take the product of $\mathcal{A}(\varphi, X_1, X_2, X_3)$ and $\mathcal{A}(\psi, X_1, X_2, X_3)$. This situation is handled by

Lemma 13 and Definition 17(h) of [3]: the automaton $\mathcal{A}(\varphi, X_1, \dots, X_m)$ has the same states as $\mathcal{A}(\varphi, X_1, \dots, X_n)$ and Properties (i) and (ii) are inherited from $\mathcal{A}(\varphi, X_1, \dots, X_n)$. \square

Proof of the Recognizability Theorem: Let φ be an MSO sentence and $\mathcal{A}(\varphi), B$ and i be constructed by the previous proposition.

Let G be a p-graph and t be any term in $T(F)$ that defines it. Then, by Property (ii), the state $q_{\mathcal{A}(\varphi)}(t)$ depends only on G (it is the same for every term t that defines G), hence can be written $q(G)$. By (i), $q(G) \in \mathcal{L}_i(B \cup \pi(G))$.

We define an equivalence relation by :

$$G \approx G' \text{ if and only if } \pi(G) = \pi(G') \text{ and } q(G) = q(G').$$

Two isomorphic graphs are equivalent by Property (ii) and two equivalent graphs have the same type. We prove that \approx is a congruence.

Let $G \approx G'$ and $H \approx H'$. Then $\pi(G \oplus H) = \pi(G' \oplus H') = \pi(G) \cup \pi(H)$.

Let t_G define G and $t_{G'}$ define G' . Then $q(G) = q_{\mathcal{A}(\varphi)}(t_G) = q(G') = q_{\mathcal{A}(\varphi)}(t_{G'})$ and similarly for H and H' . We have $q(G \oplus H) = q_{\mathcal{A}(\varphi)}(t_G \oplus t_H)$ and so, $\oplus[q_{\mathcal{A}(\varphi)}(t_G), q_{\mathcal{A}(\varphi)}(t_H)] \rightarrow_{\mathcal{A}(\varphi)} q(G \oplus H)$. Similarly, $\oplus[q_{\mathcal{A}(\varphi)}(t_{G'}), q_{\mathcal{A}(\varphi)}(t_{H'})] \rightarrow q(G' \oplus H')$ and $q(G' \oplus H') = q(G \oplus H)$ since $\mathcal{A}(\varphi)$ is deterministic. Hence, $G \oplus H \approx G' \oplus H'$. The proof is similar for all unary operations.

Since $\mathcal{L}_i(B \cup C)$ is finite for C finite, the congruence \approx has finitely many classes of each finite type $C \subseteq \mathbb{N}_+$.

A p-graph G satisfies φ if and only if $q(G)$ is an accepting state of $\mathcal{A}(\varphi)$. Hence the set of finite models of φ is a union of classes of \approx , hence is recognizable. \square

In [3], we have constructed FA for other basic properties than $X_i \subseteq X_j$, $Sgl(X_i)$ and $edg(X_i, X_j)$, and in particular, for $Card_p(X_1)$ (X_1 has p elements), $Partition(X_1, \dots, X_m)$ (X_1, \dots, X_m is a partition of the vertex set), $Path(X_1, X_2)$ (X_1 consists of two vertices linked by a path having its vertices in X_2), connectedness and existence of cycles. These FA satisfy Properties (i) and (ii) : the proofs are the same as for $X_i \subseteq X_j$, $Sgl(X_i)$ and $edg(X_i, X_j)$. However, the minimal syntax for MSO formula that we use is enough to prove the Recognizability Theorem.

The construction of FA for the properties $Card_{p,q}(X_1)$ (X_1 has p modulo q elements) yields the proof of the Recognizability Theorem for *counting monadic second-order* logic. See [6] for details.

In [3], we have also constructed "smaller" FA that work correctly on terms in $T(F)$ satisfying the special condition to be *irredundant* (no edge is created between two vertices x and y if there exists already one). For an automaton on irredundant terms, the state reached at some node u of a term t does not depend only on the graph $val(t)/u$ but also, implicitly, on the context of u in t . These automata are useful for model-checking because they are smaller than the equivalent general ones and terms can be preprocessed appropriately, but they may not satisfy Property (ii). Hence, they cannot be used in the above proof of the Recognizability Theorem.